Laminar-Turbulent Transition in Flow of Pseudoplastic Fluids with Yield Stresses

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An analysis of the laminar-turbulent transition behavior of power law non-Newtonian fluids with yield stresses reveals that considerable interaction occurs between the characterizing parameters m and Hedstrom number for low power law indexes, m. In particular, for $m < 0.4$ small yield stresses tend to destabilize the flow whereas large ones stabilize it. The results of this analysis suggest that in order to obtain correct predictions and interpretations of transitional and turbulent flows of non-Newtonian fluids, a correct modeling of the viscous rheological behavior of the fluid is essential.

Nomenclature

A = a dimensionless parameter defined by Eq. (6)
D = pipe diameter
f = friction factor
He = Hedstrom number, defined by Eq. (17)
K = laminar-turbulent transition parameter
$\bar{K}$ = maximum value of K
m = power law index in rheological model
Re = Reynolds number, defined by Eq. (8)
r = radial coordinate
u = $v/\langle v\rangle$, dimensionless velocity
v = velocity
$\langle v\rangle$ = mean velocity
Γ = $-du/d\xi$
η = consistency index in rheological model
ξ = r/r_w
$\bar{\xi}$ = value of ξ where $K = \bar{K}$
ξ_0 = τ_0/τ_w
ρ = fluid density
σ = dimensionless parameter defined by Eq. (7)
τ = shear stress
τ_0 = yield stress in rheological model

Subscripts
BP = Bingham Plastic
c = critical value
w = wall value

Introduction

THE transitional and turbulent flow of non-Newtonian fluids is a subject which has inspired considerable interest for many years. Dodge and Metzner[1] presented data for several polymer solutions and clay suspensions which they correlated empirically in terms of Metzner and Reed's[2] "generalized" power law technique. This technique has become more or less accepted as an engineering approximation in dealing with turbulent flows of non-Newtonian fluids (see Skelland[3] for example). However, more recent work (Elata et al.,[4] Wells,[5] Meyer,[6] Ernst,[7] and Van Driest[8]) has shown that more than this simple expediency is required to correlate the data for certain fluids. In addition, Hanks and Christiansen[9] showed that the Metzner-Reed technique, when used together with the Ryan-Johnson[10] transition correlation, led to grossly inaccurate predictions of the critical transition friction factor for fluids possessing yield stresses. More recently Hanks and Pratt[11] showed that the transition friction factors and Reynolds numbers for a large number of non-Newtonian slurries having Bingham plastic rheology could be well correlated by Hanks'[12] transition parameter theory, the key being proper rheological characterization of the slurries. Hanks and Dadia[13] incorporated the results of Hanks and Pratt[11] into Hanks'[14] modified mixing length turbulence model to correlate both transitional and turbulent flow of Bingham plastic slurries.

Partial attempts at correlating power-law non-Newtonian turbulent flow using mixing length theories have been presented by Clapp[15] and Hecht[16] but these are not as complete as the Hanks-Dadia[13] analysis of Bingham plastics slurries. The present authors have developed a more complete analysis[17] which will be published in another paper to appear soon. In connection with this new analysis of non-Newtonian transitional and turbulent flow, it became evident that the effects of both non-Newtonian viscosity, as approximated by a power-law type of behavior, and yield stresses were of importance. In order to describe these characteristics and their effects on turbulence, it was necessary that transition critical Reynolds numbers be known for a rheological model possessing both power-law viscous behavior and a yield stress. The purpose of the present paper is to present the details of the calculation of such critical Reynolds numbers. In the analysis presented here, the rheological model of Herschel and Bulkley[18] is used in the Hanks[12] transition parameter theory to compute critical values of a generalized Reynolds number. The theory is cast into such a form that it contains both the simple power law and Bingham plastic results as special cases. The present treatment shows very clearly the sensitivity of transitional flow to the presence of a small yield stress.

Analysis

Consider the laminar shearing flow in a pipe of a viscous non-Newtonian fluid whose rheological behavior is characterized by the Herschel-Bulkley[18] model

$$\tau = \tau_0 + \eta\left(-\frac{dv}{dr}\right)^m \qquad \tau > \tau_0 \tag{1a}$$

$$0 = -dv/dr \qquad \tau \leq \tau_0 \tag{1b}$$

where τ_0 is a yield stress, η is a power-law consistency parameter, and m is a power-law flow behavior index, usually less than unity. If $m = 1$, Eqs. (1a) and (1b) reduce to the simple linear Bingham plastic model, while for $\tau_0 = 0$, they reduce to the familiar power law.

If one defines the dimensionless variables $\xi = r/r_w$, $u =$

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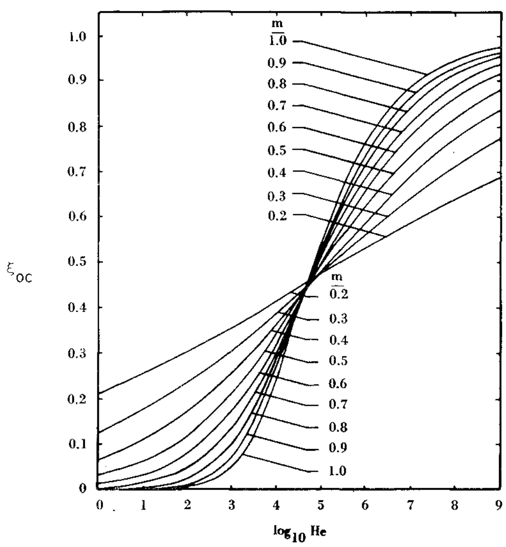

Fig. 1 ξ_{0c} as a function of *He* for various values of *m* calculated from Eq. (16).

$v/\langle v\rangle$, $\xi_0 = \tau_0/\tau_w$, and $\Gamma = -du/d\xi$, Eqs. (1a) and (1b) may be written in dimensionless form as

$$\xi = \xi_0 + A\Gamma^m \qquad \xi > \xi_0 \tag{2a}$$

$$0 = \Gamma \qquad \xi \le \xi_0 \tag{2b}$$

where

$$A = \eta(\langle v\rangle / r_w)^m/\tau_w \tag{3}$$

Equations (2a) and (2b) may readily be integrated to give the following expressions for the velocity distribution

$$u = \frac{mA^{-1/m}}{1+m}[(1-\xi_0)^{1+1/m} - (\xi-\xi_0)^{1+1/m}] \qquad \xi > \xi_0 \tag{4a}$$

$$u = u_0 = \frac{mA^{-1/m}}{1+m}(1-\xi_0)^{1+1/m} \qquad \xi \le \xi_0 \tag{4b}$$

In terms of the above dimensionless variables, the expression for the mean velocity $\langle v\rangle$ becomes

$$\int_{\xi_0}^{1} \xi^2\Gamma d\xi = 1 \tag{5}$$

where Γ is given by Eq. (2a). Evaluation of the integral in Eq. (5) gives the result

$$A = \sigma\left(\frac{m}{1+3m}\right)^m \tag{6}$$

where

$$\sigma = (1-\xi_0)^{1+m}\left[(1-\xi_0)^2 + 2\xi_0(1-\xi_0)\left(\frac{1+3m}{1+2m}\right) + \xi_0^2\left(\frac{1+3m}{1+m}\right)\right]^m \tag{7}$$

Equation (6) is a dimensionless expression for the average velocity $\langle v\rangle$ but it is not especially convenient. It is more convenient to recast this equation into terms of the familiar friction factor.

The definition of Reynolds number in non-Newtonian flow is arbitrary. We shall here employ the convention that $fRe = 16$, where $f = 2\tau_w/\rho\langle v\rangle^2$ is the Fanning friction factor. From this condition and Eqs. (3) and (6) one may easily show that

$$16/f = Re = 8\rho r_w^m\langle v\rangle^{2-m}\left[\left(\frac{m}{1+3m}\right)\right]^m \sigma/\eta \tag{8}$$

with σ being given by Eq. (7). Since $\sigma(\xi_0 = 0) = 1$ it is easily seen that Eq. (8) reduces the Metzner-Reed "generalized" Reynolds number for power-law fluids. For the limit $m = 1$ it is easily shown that

$$\sigma(m=1) = 1 - 4/3\xi_0 + 1/3\xi_0^4 \tag{9}$$

the familiar[3] Buckingham factor. In this limit Eq. (8) reduces to

$$Re(m=1) = Re_{BP}\sigma(m=1) \tag{10}$$

where $Re_{BP} = D\langle v\rangle\rho/\eta_{BP}$ and η_{BP} is the plastic viscosity or coefficient of rigidity of the Bingham plastic rheological model. Since Re_{BP} is the Reynolds number used by Hanks and Pratt,[11] the values of Re ($m = 1$) computed herein will differ from theirs by the factor $\sigma(m = 1)$.

The computation of the laminar-turbulent transition values of Re and ξ_0 will follow the method developed by Hanks.[12] In this method we use a local parameter defined by the relation

$$K = |\rho v x w|/|\nabla\cdot\tau| \tag{11a}$$

where $\mathbf{v}$ is the velocity vector, $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity vector, and τ is the deviatoric stress tensor. This parameter has been shown[12] to be very effective in predicting quantitatively the laminar-turbulent transition for both Newtonian and non-Newtonian flows in several different geometry ducts. This parameter has the following properties:

1) It is local. That is, $K = K(\xi)$.
2) $K = 0$ on solid boundaries (where $\mathbf{v} = 0$) and at points where $\mathbf{w} = 0$.
3) $K \ge 0$.
4) K possesses a maximum value at some point in the field (at $\xi = \bar{\xi}$, $K = \bar{K}$).
5) When $\bar{K} \ge 404$ (this value gives rise to $Re_c = 2100$ for Newtonian pipe flow), the flow is unstable and transition normally occurs.

In terms of the present dimensionless variables for rectilinear pipe flow, Eq. (11a) becomes

$$K = \frac{Re}{16}u\Gamma \tag{11b}$$

Re_c, the transition critical Reynolds number, is given[12] by Eq. (11a) when one sets $\xi = \bar{\xi}$ and $\bar{K} = 404$. The value of $\bar{\xi}$ is found by setting $dK/d\xi = 0$. In the present case it is the root of the following equation:

$$0 = \left.\frac{dK}{d\xi}\right|_{\bar{\xi}} = A_c^{-2/m}(\bar{\xi}-\xi_{0c})^{(1/m)-1}\left\{\frac{1}{1+m}\left[(1-\xi_{0c})^{1+1/m} - (\bar{\xi}-\xi_{0c})^{1+1/m}\right] - (\bar{\xi}-\xi_{0c})^{1+1/m}\right\} \tag{12}$$

Although $\bar{\xi} = \xi_{0c}$ is a root of Eq. (12), it would give $K = 0$, which is not useful. Thus, the root obtained by equating the expression in braces to zero is the desired value. This result is

$$\bar{\xi} = \xi_{0c} + (1-\xi_{0c})[1/(2+m)]^{m/(1+m)} \tag{13}$$

If one solves Eq. (11a) for Re, sets $\bar{K} = 404$, introduces Eqs. (2a), (4a), (6), and (13), one can obtain (after some algebra) the following expression for Re_c:

$$Re_c = [6464m/(1+3m)^2](2+m)^{(2+m)/(1+m)}\sigma_c^{2/m}/(1-\xi_{0c})^{1+2/m} \tag{14}$$

where σ_c is given by Eq. (7) with $\xi_0 = \xi_{0c}$.

All that remains to be done is calculate ξ_{0c}, the critical value of ξ_0. To accomplish this we recall that Eq. (6), which describes the laminar flow, is also valid at $Re = Re_c$. By using Eq. (3) and the definitions of f and Re, and by eliminating $\langle v\rangle$, one may rearrange Eq. (6) into the following equivalent form

$$Re_c = 2He\left[\frac{m}{1+3m}\right]^2[\sigma_c^{2/m}/\xi_{0c}^{(2/m)-1}] \tag{15}$$

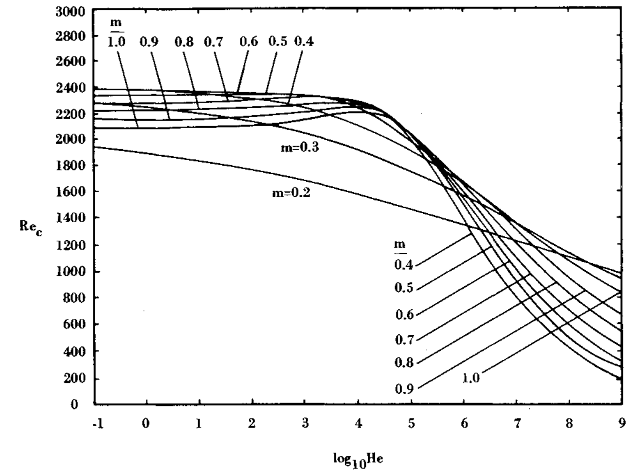

Fig. 2 ***Re_c* as function of *He* for various values of *m*.**

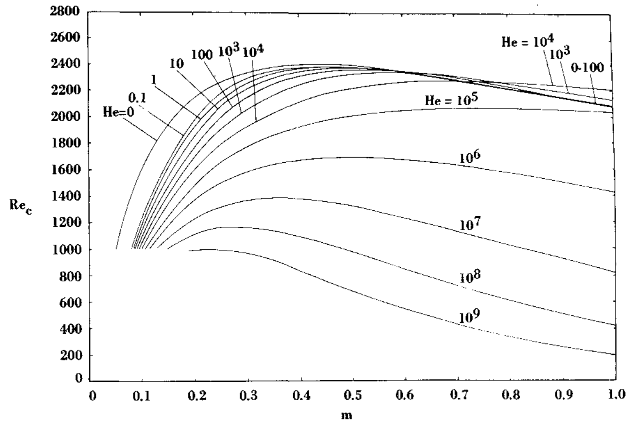

Fig. 3 ***Re_c* as a function of *m* for various values of *He*.**

Now, by eliminating Re_c between Eqs. (14) and (15) one easily obtains

$$\xi_{0c}^{(2/m)-1}/(1-\xi_{0c})^{(2/m)+1} = mHe[1/(2+m)]^{(2+m)/(1+m)}/3232 \quad (16)$$

which uniquely defines ξ_{0c} as a function of the dimensionless parameter

$$He = \frac{D^2\rho}{\tau_0}(\tau_0/\eta)^{2/m} \quad (17)$$

The parameter He is a generalization of the familiar Hedstrom number of Bingham plastic flow and reduces thereto in the limit $m = 1$.

The computation scheme is now evident. One specifies arbitrary values of m and He (corresponding to some particular fluid-pipe combination) and solves Eq. (16) for ξ_{0c}. Having this value one then uses Eq. (14) to compute Re_c. The critical friction factor is finally obtained from $f_c = 16/Re_c$ to complete the calculation.

Results and Discussion

The above calculation procedure was carried out for a series of values of m and He and the results are plotted in Figs. 1-3. In order to interpret the curves shown in these figures, it is helpful to consider the limiting forms of some of the equations as determined above. Equation (8) shows that in the limit $\xi_0 = 0$ the results should compare directly with previous results for power law fluids.[9,10,14] The curve for $He = 0$ in Fig. 3 is just the power law result which has been extensively studied and experimentally verified.[9,19] The $m = 1$ limit, as shown by Eqs. (9) and (10), does not become that used previously by Hanks and Pratt.[11] The reason for this is that these latter authors chose not to force $f = 16/Re$. Hence, their definition of Re is not fully compatible with the present choice, the two differing by the factor $\sigma(m = 1)$ given in Eq. (9). As a consequence of this difference, the curves of Re_c as function of m and He, as given in Figs. 2 and 3, will differ in qualitative shape from the corresponding curve given by Hanks and Pratt[11] for the $m = 1$ case. However, since those previous results have been well verified experimentally,[11] the present $m = 1$ curves may be considered as well established even though their form differs.

Equations (16) and (17), which define ξ_{0c} reduce directly to Hanks and Pratts[11] results in the limit $m = 1$. Consequently, the present $m = 1$ curve given in Fig. 1 is identical with theirs.

The curves in Fig. 1 reveal several interesting qualitative features of this model. First, it is apparent that the effects of pseudoplasticity and yield stresses are strongly interactive. Essentially, the parameter He reflects the influence of the yield stress τ_0, while the parameter m accounts for the pseudoplasticity. From Eq. (17) it can be seen that low values of m tend to accentuate the influence of τ_0 in determining the numerical magnitude of He. Thus, for low values of m, a low value of He corresponds to a very small value of τ_0. With this in mind, the low He part of Fig. 1 becomes quite interesting in that it shows the pronounced influence of a very small yield stress on the relative magnitude of τ_{wc} (and hence f_c). Since He is small, τ_0 must be small, but the relatively sizeable values of $\xi_{0c} = \tau_0/\tau_{wc}$ for $m \le 0.4$ suggest that low values of τ_{wc} and hence f_c should characterize highly non-Newtonian fluids with even small yield values. For this reason, the Metzner-Reed[2] method of fitting a variable parameter power law to non-Newtonian systems having low m-values is risky since it ignores any yield values. This was very graphically demonstated by Hanks and Christiansen[9] (see their Fig. 9) where errors of several hundred percent were shown to occur when the Metzner-Reed method was used. The present results show clearly the basis for that experimental observation.

For large He values, Fig. 2 shows that the effect of the pseudoplasticity is reversed. That is, the value of τ_{wc} relative to τ_0 is increased over the $m = 1$ value as m is decreased. Thus, the introduction of pseudoplasticity into the rheological behavior of a high yield stress fluid appears to stabilize the laminar flow relative to the simple Bingham plastic case. There appears to be a range of He around $4(10^4)$ for which the value of m has essentially no effect on the behavior of the flow. This occurs at about the point where the nonlinearity in the factor $\sigma(m = 1)$ becomes significant.

Figures 2 and 3 are alternative ways of displaying the effects of the parameters m and He on Re_c. From Fig. 2 it appears that the effect on He on Re_c is minimal for $He < 10^4$ and $m \ge 0.5$. This can also be seen from Fig. 3. However, both figures show a marked effect for $He > 10^4$ and $m < 0.5$. This again emphasizes the considerable interaction between m and He for low m values. It is hoped that reliable data may be developed to test the present theoretical results.

From the present results it is evident that the interaction of various types of rheological behavior can have significant influence upon the dynamic response of a fluid. One might therefore anticipate a similar circumstance when considering the problems of turbulent flows of non-Newtonian fluids. It appears that proper formulation of the rheological model is of more importance in transitional and turbulent flows in pipes than in laminar flows. This is a result which might not have been intuitively obvious. The full significance of this result is not evident without a discussion of turbulent flow. This will be the subject of a subsequent paper.

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